

Small deformations of extreme five dimensional Myers-Perry black hole initial data

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Abstract

We demonstrate the existence of a one-parameter family of initial data for the vacuum Einstein equations in five dimensions representing small deformations of the extreme Myers-Perry black hole. This initial data set has ‘ $t - \phi^i$ ’ symmetry and preserves the angular momenta and horizon geometry of the extreme solution. Our proof is based upon an earlier result of Dain and Gabach-Clement concerning the existence of $U(1)$ -invariant initial data sets which preserve the geometry of extreme Kerr (at least for short times). In addition, we construct a general class of transverse, traceless symmetric rank 2 tensors in these geometries.

1 Introduction

Einstein’s equations admit an initial-value formulation, with Cauchy data specified by the triple (Σ, h, K) where Σ is a Riemannian manifold equipped with a metric tensor h and K represents the second fundamental form of Σ , regarded as a spacelike hypersurface of spacetime. The field equations, together with the Gauss-Codazzi equations, impose the constraints

$$\begin{aligned} R_h - K^{ab}K_{ab} + (\text{tr}K)^2 &= 8\pi\mu \\ \nabla^b (K_{ab} - \text{tr}Kh_{ab}) &= -4\pi j_a \end{aligned} \tag{1}$$

where R_h is the scalar curvature of (Σ, h) and (μ, j) are the local energy density and momentum current respectively. The complete proof that solutions to the constraints evolve

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into a unique maximal development, for sufficient regularity, is a significant achievement (see e.g. [1] and for a concise summary [2]). It is an important problem to actually construct initial data with desired properties. This involves identifying the freely specifiable ‘degrees of freedom’ and then determining whether a corresponding solution exists and is unique.

A useful approach to achieve this is the conformal method ([3, 4, 5]). In the special case of data with constant mean curvature ($\text{tr}K = \text{const}$) the problem reduces to solving a conformally invariant system of equations for the conformal factor and a vector field which generates the extrinsic curvature. For spatially closed and asymptotically Euclidean initial data sets, one can prove existence using the conformal method [4] (for spacetime dimension $D \geq 4$). Subsequently, Maxwell [6] constructed asymptotically Euclidean initial data with apparent horizon boundary conditions (in particular, he treated the case with multiple disconnected apparent horizons). This case is naturally relevant to black holes.

While the above results are powerful in their generality, one can also consider the existence of initial data with very specific geometrical properties. This paper will be concerned with initial data sets which have one Euclidean end and one cylindrical end. Roughly, the latter means an initial data set (Σ, h) has an asymptotic end which is diffeomorphic to $\mathbb{R} \times N$ where N is a compact manifold. A systematic analysis of initial data on manifolds with cylindrical ends was performed in [7, 8]. In particular, existence of solutions of Lichnerowicz’s equation is proved using the powerful barrier method [9]. The purpose of our analysis, however, is to prove the existence of a rather specific class of perturbed initial data with additional properties (e.g. preserving angular momenta of the background data). We will make clear at the end of this section how our results are related to [7, 8].

Initial data sets with cylindrical ends arise within the context of stationary, extreme black holes. Extreme black holes with degenerate Killing horizons have vanishing surface gravity $\kappa = 0$, and in the limit as one approaches the horizon, Einstein’s equations decouple in a precise manner into a set of equations defined only on the horizon [10]. This gives rise to the notion of a *near-horizon geometry*, which is often thought of as an infinite ‘throat’ region in the spacetime (indeed the proper length to a spatial section of the horizon is infinite).

Extreme black holes have attracted a great deal of interest in recent years. Due to the decoupling described above, classifying near-horizon geometries is tractable and yields important information on the full space of extreme solutions (e.g. allowed geometries and topologies of spatial cross sections). Furthermore, extreme black hole geometries saturate a number of geometric inequalities which must hold for initial data sets and for marginally outer trapped surfaces in four dimensions [11, 12, 13] (see also [14] for work on the latter problem in $D > 4$). Finally, extreme black holes have the simplest microscopic description within string theory, and so are an important testing ground for various calculations in quantum gravity, the most well-known of which is black hole entropy counting. Recently, due to the work of Aretakis and others [15, 16, 17, 18, 19], extreme black holes have been shown to be unstable to a certain horizon instability. An

alternative approach to studying the non-linear instability of the extreme Kerr-Newman family using perturbations of the initial data of extreme Reissner-Nordström also has recently appeared [20].

A spacelike slice of such a near-horizon geometry has the form of the geometry of a cylindrical end, where $N \cong H$, a spatial cross-section of the horizon. Hence initial data for an asymptotically flat extreme black hole has one asymptotic Euclidean region and an asymptotically cylindrical end. The simplest example of this occurs for initial data of the extreme $M = \sqrt{J}$ Kerr black hole [5]. These authors, using the conformal method alluded to above, proved that there exists a one-parameter family of axisymmetric initial data of the vacuum Einstein equations which preserve the asymptotic behaviour, angular momenta, and area of the cylindrical end (this area corresponds to the area of the spatial sections of the horizon of the Kerr black hole). In particular, as a consequence of the geometric inequalities, one can show the energy of any member of this family must be strictly greater than that of the extreme Kerr initial data. Note that the solutions satisfy weak regularity conditions (i.e. they belong to a certain Sobolev space) and in particular are not generically smooth, let alone analytic. This last distinction could be important when considering the evolution of this initial data. The extreme Kerr black hole is known to be the unique (analytic) vacuum, stationary, rotating asymptotically flat spacetime containing a single degenerate horizon [21][22, 23]. Hence the evolution of the initial data sets discussed above could settle down to non-analytic asymptotically flat (possibly stationary) extreme black holes. Of course, we cannot address this issue without understanding the evolution.

It is natural to investigate the possibility of extending the result of [5] to extreme, five-dimensional black holes. The simplest candidate would be extreme Myers-Perry black hole [24], which is qualitatively similar to Kerr. A maximal slice can be found with $U(1)^2$ isometry and has topology $\mathbb{R} \times S^3$ [25]. However there are two main differences as one moves from $n = 3$ to $n = 4$ spatial dimensions. First, it turns out we will have to construct solutions of the constraint equations which belong to Bartnik's weighted Sobolev spaces $W_\delta'^{k,p}$ [26]. Our asymptotic fall-off conditions at the Euclidean end and cylindrical end require $kp > n$ (see Lemma A.1 in [5]). We only require weak differentiability to second order, so we take $(k, p, \delta) = (2, 3, -1)^1$ whereas in the analysis of [5], $(k, p, \delta) = (2, 2, -1/2)$. The latter spaces are weighted Hilbert spaces, which prove extremely useful in the elegant construction given in [5]. Second, we require five scalar functions to characterize our data as opposed to two and our geometries have $U(1)^2$ symmetry which complicates the parameterization of the extrinsic curvature. Our main result is Theorem 3.1 and it can be informally stated as follows:

There exists a one parameter, $U(1)^2$ -invariant, maximal family of solutions to Einstein's constraint equations. This family of data is second order differentiable with respect to an appropriate norm and it has the same angular momentum and area of the event horizon

¹One could also take $(k, p, \delta) = (3, 2, -1)$ but this leads to a stronger regularity condition for a particular elliptic operator and the functions in the background metric do not satisfy this regularity.

of an extreme Myers-Perry black hole. Moreover, the geometry of this family is close (in a suitable sense) to the extreme Myers-Perry initial data set.

It is important to clarify what is new about this result and how it is related to the analysis of [7, 8]. In particular, Theorem 6.1 of [7] asserts the existence of a class of solutions to Lichnerowicz's equation for complete initial data with non-negative scalar curvature and strictly positive scalar curvature on cylindrical ends. These results are quite powerful and general in that no symmetry assumptions are made on the data. However, if one wishes to impose additional conditions (e.g. axisymmetry) on the data, one might be interested if there exists special families of data with the same ADM energy, conserved angular momenta and/ or area of the cylindrical end. This work is concerned with finding a class of initial data suitably close to the extreme Myers-Perry data that preserves the angular momenta and area of its cylindrical end. This data can be interpreted as perturbations of extreme Myers-Perry. To prove this result, we need to first consider a more general problem of finding transverse, traceless symmetric rank 2 tensors on $U(1)^2$ -invariant geometries which generalizes [27]. To the best of our knowledge, this work has not appeared before and should be useful in various contexts when considering initial data with symmetries.

This paper is organized as follows. In Section 2 we discuss the maximal slices of the extreme Myers-Perry solution. Section 3 states our theorem and Section 4 provides most of the technical details in the proof. We conclude with a discussion. The appendices collects a number of useful theorems which we use in our proof, and some technical properties of the Myers-Perry solution which we use to establish our result.

2 Initial Data with $U(1)^2$ symmetry

In this work we consider general initial data sets (Σ, h, K) which are invariant under $U(1)^2$ isometry. In addition we will restrict attention to maximal slices, i.e. $\text{tr}K = 0$. Finally, as explained in detail below, the Myers-Perry maximal initial data set of interest has a further useful property (' $t - \phi^i$ ' symmetry) and we will impose this on our class of initial data sets as well. In the following for convenience we will simply refer to initial data satisfying these various conditions as 'biaxisymmetric'.

2.1 Extreme Myers-Perry black hole and initial data

Our starting point is the five-dimensional vacuum Myers-Perry black hole (M, g) with metric [28]

$$\begin{aligned}
g = & -dt^2 + \frac{\mu}{\Sigma} \left(dt + a \sin^2 \theta d\varphi + b \cos^2 \theta d\psi \right)^2 + \frac{\tilde{r}^2 \Sigma}{\Delta(\tilde{r})} d\tilde{r}^2 + \Sigma d\theta^2 \\
& + (\tilde{r}^2 + a^2) \sin^2 \theta d\varphi^2 + (\tilde{r}^2 + b^2) \cos^2 \theta d\psi^2
\end{aligned} \tag{2}$$

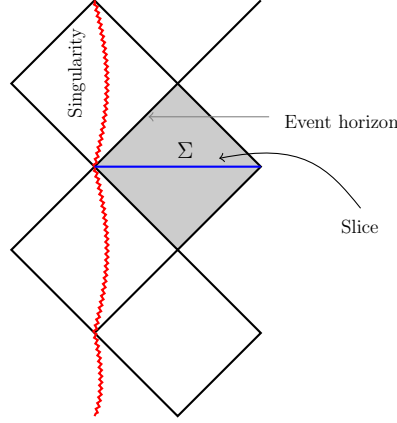


Figure 1: Carter-Penrose diagram of extreme Myers-Perry black hole. The gray region is domain of outer communication=DOC

where

$$\Sigma = \tilde{r}^2 + b^2 \sin^2 \theta + a^2 \cos^2 \theta, \quad (3)$$

$$\Delta(\tilde{r}) = (\tilde{r}^2 + a^2)(\tilde{r}^2 + b^2) - \mu \tilde{r}^2. \quad (4)$$

The solution is parameterized by (μ, a, b) with orthogonally transitive isometry group $\mathbb{R}_t \times U(1)^2$, where \mathbb{R} is the time translation symmetry and $U(1)^2$ is the rotational symmetry generated by ∂_ψ and ∂_ϕ . Here (\tilde{r}, θ) parameterize the two-dimensional surfaces orthogonal to orbits of the isometry group. We take $\mu > 0$ so that the mass of the spacetime $M > 0$ and without loss of generality we take $a, b > 0$. The horizons of this black hole are located at the roots of $\Delta(\tilde{r})$, denoted $\tilde{r}_{H\pm}$. The metric is written in a chart that covers the black hole exterior $\tilde{r}_{H+} < \tilde{r} < \infty$. In addition $0 < \theta < \pi/2$, and ψ, ϕ are periodic with period 2π . As is well known, the solution is qualitatively similar to the Kerr solution. In the extreme limit, $\mu = (a + b)^2$ and $\Delta(\tilde{r}) = (\tilde{r}^2 - ab)^2$. We define a new radial coordinate $r^2 = \tilde{r}^2 - ab$, resulting in

$$g = -dt^2 + \frac{\mu}{\Sigma} (dt + a \sin^2 \theta d\varphi + b \cos^2 \theta d\psi)^2 + \frac{\Sigma}{r^2} dr^2 + \Sigma d\theta^2 + (r^2 + ab + a^2) \sin^2 \theta d\varphi^2 + (r^2 + ab + b^2) \cos^2 \theta d\psi^2. \quad (5)$$

where $r > 0$. There is a degenerate Killing horizon located at $r = 0$, which can be seen by transforming to an adapted Gaussian coordinate system. The mass and angular momenta are easily evaluated using Komar integrals, giving

$$M = \frac{3\pi}{8} \mu, \quad J^\varphi = \frac{\pi \mu a}{4}, \quad J^\psi = \frac{\pi \mu b}{4}. \quad (6)$$

The extremality condition can be written

$$M^3 = \frac{27\pi}{32} (J^\psi + J^\varphi)^2 \quad (7)$$

Consider a spacelike hypersurface Σ corresponding to a $t = \text{constant}$ slice in the above geometry. The induced metric and extrinsic curvature Σ is easily found to be

$$h = \frac{\Sigma}{r^2} dr^2 + \Sigma d\theta^2 + \left(\frac{a^2 \sin^2 \theta \mu}{\Sigma} + (r^2 + ab + a^2) \right) \sin^2 \theta d\varphi^2 + 2 \frac{ab \cos^2 \theta \sin^2 \theta \mu}{\Sigma} d\varphi d\psi + \left(\frac{b^2 \cos^2 \theta \mu}{\Sigma} + (r^2 + ab + b^2) \right) \cos^2 \theta d\psi^2 \quad (8)$$

$$K = - \frac{a\mu(r^2 + ab + b^2)(\Sigma + r^2 + ab + a^2)}{\Sigma^2 \sqrt{g^{tt}} r^3} \sin^2 \theta dr d\varphi + \frac{a\mu(a^2 - b^2) \cos \theta \sin^3 \theta}{\Sigma^2 \sqrt{g^{tt}}} d\theta d\varphi - \frac{b\mu(r^2 + ab + a^2)(\Sigma + r^2 + ab + b^2)}{\Sigma^2 \sqrt{g^{tt}} r^3} \cos^2 \theta dr d\psi + \frac{b\mu(a^2 - b^2) \cos^3 \theta \sin \theta}{\Sigma^2 \sqrt{g^{tt}}} d\theta d\psi \quad (9)$$

Although not time-symmetric, this initial data has in addition ' $t - \phi^i$ ' symmetry (under the simultaneous diffeomorphisms $(\varphi, \psi) \rightarrow (-\varphi, -\psi)$ h is invariant and K reverses sign) [29]. This symmetry in particular implies $\text{tr} K = 0$, i.e. the slices are maximal. The triple (Σ, h, K) forms a vacuum maximal initial data set (i.e. a solution of (1) with $\mu = j = 0$) for the extreme black hole exterior:

$$R_h - K^{ab} K_{ab} = 0 \quad \nabla^b K_{ab} = 0 \quad (10)$$

The pair (Σ, h) represents a Riemannian manifold with one asymptotically flat end and one asymptotically cylindrical end. Σ is diffeomorphic to $\mathbb{R} \times S^3 \cong \mathbb{R}^4 \setminus \{0\}$ [25] and the spatial metric (8) is a cohomogeneity two, asymptotically flat metric that extends globally onto Σ . The metric has the following fall off conditions at its asymptotically flat end:

$$h_{ab} = \delta_{ab} + \mathcal{O}(r^{-2}), \quad \partial h_{ab} = \mathcal{O}(r^{-3}) \quad (11)$$

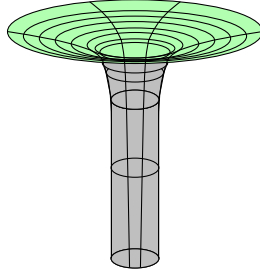
$$K_{ab} = \mathcal{O}(r^{-4}), \quad \partial K_{ab} = \mathcal{O}(r^{-5}) \quad (12)$$

where δ is the Euclidean metric on \mathbb{R}^4 . To investigate the geometry of (8) as $r \rightarrow 0$, perform the transformation $s = -\ln r$. This reveals a new asymptotic region corresponding to the limit $s \rightarrow \infty$ with geometry

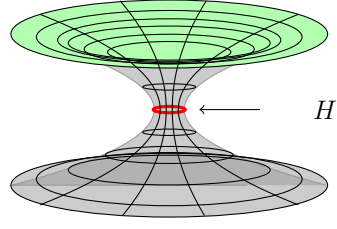
$$h = \sigma(\theta) \left[ds^2 + d\theta^2 + \frac{(a+b)^2}{\sigma(\theta)^2} \left(a^2 \sin^2 \theta d\varphi^2 + b^2 \cos^2 \theta d\psi^2 + ab [\cos^2 \theta d\varphi + \sin^2 \theta d\psi]^2 \right) \right] \quad (13)$$

$$K = \frac{\mu \sqrt{ab} (a \cos^2 \theta + b \sin^2 \theta + a)}{\sqrt{(a \cos^2 \theta + b \sin^2 \theta)^3 (a+b)^3}} \sin^2 \theta ds d\varphi + \frac{\mu \sqrt{ab} (a \cos^2 \theta + b \sin^2 \theta + b)}{\sqrt{(a \cos^2 \theta + b \sin^2 \theta)^3 (a+b)^3}} \cos^2 \theta ds d\psi$$

It can be shown that the (θ, φ, ψ) part of the metric can be globally extended to an inhomogeneous metric on S^3 [30]. Thus as $r \rightarrow 0$ (Σ, h) has a cylindrical end with geometry $\mathbb{R} \times S^3$. A schematic diagram of the slice (Σ, h) is given in Figure 2(a).



(a) Spatial slice of an extreme black hole with one cylindrical end and one asymptotic end.



(b) Spatial slice of a non-extreme black hole with two asymptotic ends.

Figure 2: Slice of Myers-Perry solution in extreme and usual cases

2.2 Biaxisymmetric Initial Data

Our goal is to construct initial data which represent deformations of the Myers-Perry initial data discussed above. The strategy, following [31], is to use the conformal method to reduce the problem to an elliptic PDE for a single scalar field. First however, we will need to parameterize our initial data sets appropriately, to isolate the functional degrees of freedom. For the biaxisymmetric data sets under consideration, there is a convenient way to achieve this, which generalizes the approach for axisymmetric three-dimensional initial data sets.

Consider an asymptotically flat spacetime with an isometry group which admits an $U(1)^2$ subgroup. We now briefly review the computation of angular momenta from the twist potentials (see [14] for a general discussion with $U(1)^{D-3}$ rotational symmetries). We will denote the generators of rotational symmetries as m_i , $i = 1, 2$. The orbits of the m_i have period 2π . For simplicity, assume there is one asymptotic end and let S_∞^3 represent the sphere at spatial infinity. We will take n and s to be unit timelike and spacelike vector fields which span the tangent space normal to S_∞^3 . As is well known, one can define conserved angular momenta from the Komar integrals

$$J_i = \frac{1}{16\pi} \int_{S_\infty^3} \star dm_i = \frac{1}{16\pi} \int_{S_\infty^3} i_n i_s dm_i dS \quad (14)$$

where dS represents the volume element on S_∞^3 . The vacuum equations imply the following one-forms are closed:

$$\lambda_i = \star(m_1 \wedge m_2 \wedge dm_i) \quad (15)$$

and we may therefore define local twist potentials ω_i satisfying $d\omega_i = \lambda_i$. We assume that the spacetime is simply connected so the ω_i are globally defined. We can then evaluate the J_i in terms of these twist potentials as follows. Using the fact the m_i generate commuting isometries and applying an interior derivative with respect to m_i ,

we find

$$\begin{aligned} dm_i &= \frac{1}{\det g_{ij}} [i_{m_1} i_{m_2} \star \lambda_i + (g(m_1, m_2) dg_{2i} - g(m_2, m_2) dg_{1i}) m_1 \\ &+ (g(m_1, m_2) dg_{1i} - g(m_1, m_1) dg_{2i}) m_2] \end{aligned} \quad (16)$$

where $g(m_i, m_j) = m_i \cdot m_j$ and $\det g_{ij} = g(m_1, m_1)g(m_2, m_2) - (g(m_1, m_2))^2$. Thus we find

$$i_n i_s dm_i = \frac{i_n i_s i_{m_1} i_{m_2} \star d\omega_i}{\det g_{ij}} \quad (17)$$

Define the one-form

$$\xi \equiv (\det g_{ij})^{-1/2} \star (n \wedge s \wedge m_1 \wedge m_2) \rightarrow \xi_a = (\det g_{ij})^{-1/2} \epsilon_{abcde} n^b s^c m_1^d m_2^e \quad (18)$$

It can be checked that ξ has unit length and is orthogonal to remaining members of the co-frame (n, s, m_i) (and in particular is tangent to S_∞^3). We may then define a coordinate x such that ξ is proportional to dx , i.e. $\xi = \sqrt{g_{xx}} dx$. As discussed in precise detail in [14], x parameterizes $S_\infty^3/U(1)^2$ and we normalize it so $-1 \leq x \leq 1$ where $x = \pm 1$ correspond to the poles where the m_i vanish. We then have $i_n i_s \star dm_i = (\det g_{ij})^{-1/2} \xi \cdot d\omega_i$ and so

$$J_i = \frac{1}{16\pi} \int_{S_\infty^3} (\det g_{ij})^{-1/2} (\xi \cdot (d\omega_i)) dS = \frac{\pi}{4} \int_{-1}^1 \frac{\partial \omega_i}{\partial x} dx = \frac{\pi}{4} (\omega_i(1) - \omega_i(-1)) \quad (19)$$

It is useful in the following to work with respect to the preferred tetrad (n, s, ξ, m_i) . For concreteness we introduce the vector fields $\eta^a = m_1^a$ and $\gamma^a = m_2^a$ with associated scalar products $\eta^a \eta_a = \eta$, $\eta^a \gamma_a = L$, $\gamma^a \gamma_a = \gamma$, and $H = \det g_{ij} = \eta\gamma - L^2$. One can write the metric in this basis as

$$g_{ab} = -n_a n_b + s_a s_b + \xi_a \xi_b + \frac{\gamma}{H} \eta_a \eta_b + \frac{\eta}{H} \gamma_a \gamma_b - \frac{2L}{H} \eta_{(a} \gamma_{b)} \quad (20)$$

In this notation, the expression (16) is

$$\bar{\nabla}^k \eta^l = \frac{1}{2H} \epsilon^{aijlk} \eta_i \gamma_j \lambda_{1a} + \frac{2}{H} \gamma_j \bar{\nabla}^{[k} \eta^{j]} P^l + \frac{1}{H} \bar{\nabla}^{[k} \eta B^l] + \frac{1}{H} \gamma_j \bar{\nabla}^j \eta \eta^{[l} \gamma^{k]} \quad (21)$$

$$\bar{\nabla}^k \gamma^l = \frac{1}{2H} \epsilon^{aijlk} \eta_i \gamma_j \lambda_{2a} + \frac{1}{H} \bar{\nabla}^{[k} \gamma P^l] + \frac{2}{H} \eta_j \bar{\nabla}^{[k} \gamma^{j]} B^l + \frac{1}{H} \gamma_j \bar{\nabla}^j \eta \eta^{[l} \gamma^{k]} \quad (22)$$

where $P^k \equiv \eta \gamma^k - L \eta^k$, and $B^k \equiv \eta^k \gamma - L \gamma^k$. As in four dimensions, we will now establish sufficient conditions under which one can construct the extrinsic curvature from potentials. We will restrict attention to initial data that are invariant under the biaxial $U(1)^2$ symmetry, i.e. (Σ, h) admits an $U(1)^2$ acting as isometries and $L_{m_i} K = 0$ where once again we denote the generators of the rotational symmetries by m_i . We are of course interested in spacetimes with $U(1)^2$ isometry, and one can always find initial data surfaces with this symmetry; conversely, given initial data with these symmetries, the evolution will preserve the symmetry, although they may be ‘hidden’ [32]. To begin,

it is easiest to introduce a tetrad for the metric on the spacelike slice such that (in the rest of this note, we will use Latin indices $a, b \dots$ to run over $1 \dots 4$)

$$h_{ab} = q_{ab} + h_{ij} m_a^i m_b^j \quad (23)$$

where $q_{ab} = s_a s_b + \xi_a \xi_b$ is the part of the metric orthogonal to the surfaces of transitivity of the $U(1)^2$ action and $h_{ij} = m_i \cdot m_j$. In terms of the spacetime tetrad presented earlier, one can think of h_{ab} as the metric induced on the surface with normal n , i.e. $h_{ab} = g_{ab} + n_a n_b$. We then have

$$h = q_{ab} + \frac{\gamma}{H} \eta_a \eta_b + \frac{\eta}{H} \gamma_a \gamma_b - \frac{2L}{H} \eta_{(a} \gamma_{b)} \quad (24)$$

Now consider a maximal slice $\text{tr}K = 0$ with a given extrinsic curvature tensor K_{ab} . The square of this tensor, which appears in the constraint equations (1), can be computed in the frame defined above:

$$\begin{aligned} K_{ab} K^{ab} &= K_{ab} K_{cd} h^{ac} h^{bd} \\ &= K_{ab} K_{cd} q^{ac} q^{bd} + (K_{ab} \eta^a \eta^b)^2 \frac{\gamma^2}{H^2} + (K_{ab} \gamma^a \gamma^b)^2 \frac{\eta^2}{H^2} + (K_{ab} \eta^a \eta^b) (K_{ab} \gamma^a \gamma^b) \frac{4L^2}{H^2} \\ &\quad + \frac{2\gamma\eta}{H^2} (K_{ab} \eta^a \gamma^b)^2 - \frac{2L\gamma}{H^2} (K_{ab} \eta^a \eta^b) (K_{cd} \eta^c \gamma^d) - \frac{2L\eta}{H^2} (K_{ab} \eta^a \gamma^b) (K_{cd} \gamma^c \gamma^d) \\ &\quad + \frac{2\gamma}{H} S_a^2 S^{2a} + \frac{2\eta}{H} S_a^1 S^{1a} - \frac{4L}{H} S_a^1 S^{2a} \end{aligned} \quad (25)$$

where the S_a^i are defined as

$$S_a^1 \equiv K_{ab} \eta^b + \frac{L}{H} \gamma_a K_{cd} \eta^c \eta^d - \frac{\gamma}{H} \eta_a K_{cd} \eta^c \eta^d + \frac{L}{H} \eta_a K_{cd} \gamma^c \eta^d - \frac{\eta}{H} \gamma_a K_{cd} \gamma^c \eta^d \quad (26)$$

$$S_a^2 \equiv K_{ab} \gamma^b + \frac{L}{H} \eta_a K_{cd} \gamma^c \gamma^d - \frac{\eta}{H} \gamma_a K_{cd} \gamma^c \gamma^d + \frac{L}{H} \gamma_a K_{cd} \gamma^c \eta^d - \frac{\gamma}{H} \eta_a K_{cd} \gamma^c \eta^d, \quad (27)$$

One can easily check that S_a^1 and S_a^2 are each orthogonal to γ^a and η^a and they are invariant under Lie derivatives with respect to γ^a and η^a . Furthermore, because K_{ab} satisfies the momentum constraint, $\nabla_a K^{ab} = 0$, one can deduce that the S^i are divergenceless; $d \star S^i = 0$. Now define the one-forms

$$\mathcal{K}^i = \star(S^i \wedge m_2 \wedge m_1) \quad (28)$$

which in our basis take the form

$$\mathcal{K}_a^1 = \epsilon_{abcd} S^{1b} \gamma^c \eta^d, \quad \mathcal{K}_a^2 = \epsilon_{abcd} S^{2b} \gamma^c \eta^d \quad (29)$$

We note these forms are closed, i.e.

$$d\mathcal{K}^i = i_{m_2} i_{m_1} d \star S^i = 0 \quad (30)$$

We then define the potentials $\bar{\omega}_i$ by

$$\mathcal{K}^i = -\frac{d\bar{\omega}_i}{2} \quad (31)$$

These $\bar{\omega}_i$ are in fact the pullback to Σ of the spacetime twist potentials ω_i defined in the previous section. The proof of this statement is similar to the three dimensional case given in [33], using the expression (21). Hence as we are working at the level of initial data we will simply use ω_i from now on. One can invert these expressions to find

$$S^i = \frac{1}{H} i_{m_2} i_{m_1} \star \mathcal{K}^i = \frac{1}{2H} i_{m_1} i_{m_2} \star d\omega^i \quad (32)$$

In our tetrad basis,

$$S_a^1 = -\frac{1}{H} \epsilon_{abcd} \gamma^b \eta^c \mathcal{K}^{1d} = \frac{1}{2H} \epsilon_{abcd} \gamma^b \eta^c d\omega^{1d} \quad S_a^2 = -\frac{1}{H} \epsilon_{abcd} \gamma^b \eta^c \mathcal{K}^{2d} = \frac{1}{2H} \epsilon_{abcd} \gamma^b \eta^c d\omega^{2d} \quad (33)$$

Now we define a symmetric, divergence free, and trace free tensor field

$$\bar{K}_{ab} := \frac{2}{H} \left[(\eta S^2_{(a} \gamma_{b)}) - L S^1_{(a} \gamma_{b)}) + (\gamma S^1_{(a} \eta_{b)}) - L S^2_{(a} \eta_{b)}) \right]. \quad (34)$$

Then the full contraction of this tensor is

$$\bar{K}_{ab} \bar{K}^{ab} = \frac{2}{H} \left[\gamma S_a^1 S^{1a} + \eta S_a^2 S^{2a} - 2L S_a^1 S^{2a} \right] \quad (35)$$

Now assume the initial data has $t - \phi^i$ symmetry as defined in [29]. Here ϕ^i are coordinates adapted to the commuting Killing fields m_i . Then: 1) $\partial/\partial\phi^i$ are Killing vector generator of $U(1)^2$ isometry group of (Σ, h_{ab}) , and 2) $\phi^i \rightarrow -\phi^i$ is a diffeomorphism which preserves h_{ab} but reverses the sign of K_{ab} . This is equivalent to the following conditions

1. $K_{ab} \eta^a \eta^b = K_{ab} \gamma^a \gamma^b = K_{ab} \gamma^a \eta^b = 0$
2. $K_{ab} q^{ac} q^{bd} = 0$

By condition (1) all terms of (25) are zero except the last three ones, also by definition of S_a^i we have

$$S_a^1 = K_{ab} \gamma^b \quad S_a^2 = K_{ab} \eta^b \quad (36)$$

Moreover, by condition (2) we have

$$\begin{aligned} 0 &= K_{ab} q^{ac} q^{bd} \\ &= K_{ab} \left(h^{ac} - \frac{\gamma}{H} \eta^a \eta^c - \frac{\eta}{H} \gamma^a \gamma^c + \frac{2L}{H} \eta^a \gamma^c \right) \left(h^{bd} - \frac{\gamma}{H} \eta^b \eta^d - \frac{\eta}{H} \gamma^b \gamma^d + \frac{2L}{H} \eta^b \gamma^d \right) \\ &= K^{cd} - \frac{2}{H} \left[(\eta S^{2(c} \gamma^{d)}) - L S^{1(c} \gamma^{d)}) + (\gamma S^{1(c} \eta^{d)}) - L S^{2(c} \eta^{d)}) \right] \\ &= K^{cd} - \bar{K}^{cd} \end{aligned} \quad (37)$$

Then we see that an arbitrary maximal $t - \phi^i$ -symmetric extrinsic curvature tensor can be constructed from the twist potentials ω_i :

$$K_{ab} = \bar{K}_{ab} \quad (38)$$

We emphasize that in general, one cannot construct the complete extrinsic curvature tensor directly from twist potentials. Although we are not going to use the following, it is interesting to see the relationship between the form of the extrinsic curvature given in (38) and the expression given in [29], valid in $t - \phi^i$ symmetry:

$$K_{ab} = J^i_{(a} \phi^i_{b)} \quad i = 1, 2 \quad (39)$$

(see also [25]). We then have

$$J_a^1 = \frac{2}{H} (\eta S_a^2 - L S_a^1), \quad J_a^2 = \frac{2}{H} (\gamma S_a^1 - L S_a^2) \quad (40)$$

3 Main result

The classical method to prove the existence of solutions to the constraint equations is the conformal method. In the case of extreme Myers-Perry with initial data we defined in section 2.1 (Σ, h_{ab}, K_{ab}) we can write the metric in the following conformal form

$$h_{ab} = \Phi_0^2 \tilde{h}_{ab}. \quad (41)$$

with

$$\tilde{h} = e^{2U} (d\rho^2 + dz^2) + \sigma'_{ij} d\phi^i d\phi^j \quad (42)$$

where $\rho = \frac{1}{2}r^2 \sin 2\theta$, $z = \frac{1}{2}r^2 \cos 2\theta$, $\det \sigma'_{ij} = \rho^2$, $\phi^1 = \varphi$, and $\phi^2 = \psi$. The conformal factor Φ_0 and the functions U and σ'_{ij} in the metric are defined and studied in Appendix B. We may also write

$$K_{ab} = \Phi_0^{-2} \tilde{K}_{ab}. \quad (43)$$

and by section 2.2, $t - \phi^i$ symmetry of the initial data implies we may express the second factor as

$$\tilde{K}_{ab} = \frac{2}{\rho^2} \left[\left(\sigma'_{22} \tilde{S}^1_{(a} \eta_{b)} - \sigma'_{12} \tilde{S}^2_{(a} \eta_{b)} \right) + \left(\sigma'_{11} \tilde{S}^2_{(a} \gamma_{b)} - \sigma'_{12} \tilde{S}^1_{(a} \gamma_{b)} \right) \right] \quad (44)$$

where $\gamma^a = \left(\frac{\partial}{\partial \varphi} \right)^a$, $\eta^a = \left(\frac{\partial}{\partial \psi} \right)^a$ and

$$\tilde{S}_a^1 = \frac{1}{2\rho^2} \tilde{\epsilon}_{abcd} \gamma^b \eta^c \tilde{\nabla}^d \omega_\varphi, \quad \tilde{S}_a^2 = \frac{1}{2\rho^2} \tilde{\epsilon}_{abcd} \eta^b \gamma^c \tilde{\nabla}^d \omega_\psi \quad (45)$$

where the twist potentials ω_i are given in Appendix B and $\tilde{\epsilon}_{abcd}$ and $\tilde{\nabla}$ are respectively the volume element and the connection associated to \tilde{h}_{ab} . This particular form of extrinsic curvature implies $\tilde{\nabla}_a \tilde{K}^{ab} = 0$ and so it satisfies (10). We will perturb about this solution (i.e. we freely specify variations of the functions appearing in the metric \tilde{h} with appropriate fall-off behaviour) and demonstrate the existence of a conformal factor Φ that solves the constraint equations, yielding a new family of initial data (Σ, h_{ab}, K_{ab}) where Φ_0 is replaced by Φ above. More precisely, our main result is

Theorem 3.1. Let (Σ, h_{ab}, K_{ab}) be the maximal biaxisymmetric initial data set of extreme Myers-Perry described in section 2.1 with angular momenta J_φ and J_ψ and mass M . Then there is a small λ_0 such that for $-\lambda_0 < \lambda < \lambda_0$ there exists a family of initial datasets $(\Sigma, h_{ab}^\lambda, K_{ab}^\lambda)$ (i.e. solutions of the constraints on Σ) such that:

1. For $\lambda = 0$ the family of initial data is that of extreme Myers-Perry initial data, i.e. (Σ, h_{ab}, K_{ab}) . The family is differentiable in λ and it is close to extreme Myers-Perry with respect to an appropriate norm which involves two derivatives of the metric.
2. The data have the same asymptotic geometry as the extreme Myers-Perry initial dataset. The angular momenta and the area of the cylindrical end in the family do not depend on λ ; they have same value as in (Σ, h_{ab}, K_{ab}) , namely J_φ , J_ψ and A_0 , respectively.
3. The family of data are biaxisymmetric and maximal (i.e $\text{tr} K^\lambda = 0$).

An important parameter of an initial data set with a cylindrical end is the area of a cross-section. If $A(r)$ is the area of constant r , we have

$$A_0 = \lim_{r \rightarrow 0} A(r) = 2\pi^2 \mu^2 \sqrt{ab}. \quad (46)$$

This corresponds to the area of the event horizon of the corresponding extreme Myers-Perry black hole. Consider a member of the family of initial data set $(\Sigma, h_{ab}^\lambda, K_{ab}^\lambda)$ for fixed $\lambda \neq 0$. By an argument similar to that given in [5], the fall-off of the lapse and shift can always be selected so that the geometry of the cylindrical end and its area will be preserved, for sufficiently short times.

4 Proof of main result

We now turn to the derivation of the result discussed in the previous section.

Proof. Let (Σ, h_{ab}, K_{ab}) be the maximal initial data set (given in Appendix C) of the extreme Myers-Perry black hole. These satisfy the constraint equations:

$$R - K_{ab} K^{ab} = 0, \quad (47)$$

$$\nabla^a K_{ab} = 0. \quad (48)$$

To construct a solution of these constraint equations we use classical conformal method with rescaling

$$h_{ab} = \Phi_0^2 \tilde{h}_{ab}, \quad K_{ab} = \Phi_0^{-2} \tilde{K}_{ab}. \quad (49)$$

where \tilde{h}_{ab} and \tilde{K}_{ab} are defined in equations (42) and (44), respectively. In conformal data the constraint equations are

$$\Delta_{\tilde{h}} \Phi_0 - \frac{1}{6} \tilde{R} \Phi_0 + \frac{1}{6} \tilde{K}_{ab} \tilde{K}^{ab} \Phi_0^{-5} = 0. \quad (50)$$

$$\tilde{\nabla}_b \tilde{K}^{ab} = 0. \quad (51)$$

By construction \tilde{K}_{ab} in section 2.2 is always divergence-free and traceless, so the momentum constraint equation (51) is automatically satisfied and we need only consider the Lichnerowicz equation (50). The Laplace operator associated with the metric (42) (for any U, σ'_{ij}) in biaxial symmetry can be written

$$\Delta_{\tilde{h}} \Phi = \frac{e^{-2U}}{r^2} \Delta_4 \Phi \quad (52)$$

Where Φ is an arbitrary function of only r and θ and Δ_4 is the flat four dimensional Laplace operator respect to metric

$$\begin{aligned} \delta_4 &= \frac{1}{2\sqrt{\rho^2 + z^2}} (d\rho^2 + dz^2) + \left(\sqrt{\rho^2 + z^2} - z \right) d\varphi + \left(\sqrt{\rho^2 + z^2} + z \right) d\psi^2 \\ &= dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi + r^2 \cos^2 \theta d\psi^2 \end{aligned} \quad (53)$$

The scalar curvature of the metric (42) is

$$\tilde{R} = e^{-2U} \left(-2\Delta_2 U + \frac{\det d\sigma'}{2\rho^2} \right) \equiv \frac{e^{-2U}}{r^2} \tilde{R}_0 \quad (54)$$

where Δ_2 is the Laplacian with respect to the flat 2 dimensional metric i.e. $\delta_2 = d\rho^2 + dz^2$.

The extrinsic curvature is

$$\tilde{K}_{ab} \tilde{K}^{ab} = \frac{e^{-2U}}{2\rho^4} \left[\sigma'_{11} (d\omega_\psi)^2 + \sigma'_{22} (d\omega_\varphi)^2 - 2\sigma'_{12} (d\omega_\varphi) \cdot (d\omega_\psi) \right] \equiv \frac{e^{-2U}}{r^2} \tilde{K}_0^2 \quad (55)$$

where \cdot is the inner product with respect to δ_2 . Then the Lichnerowicz equation (50) for the conformal triple $(\Sigma, \tilde{h}_{ab}, \tilde{K}_{ab})$ is

$$\Delta_4 \Phi_0 - \frac{\tilde{R}_0}{6} \Phi_0 + \frac{\tilde{K}_0^2}{6\Phi_0^5} = 0 \quad (56)$$

We now perturb equation (56) about the solution given by the maximal initial data for the extreme Myers-Perry black hole by taking

$$\begin{aligned} U &\rightarrow U + \lambda U_1 \\ \sigma'_{ij} &\rightarrow \sigma'_{ij} + \lambda \bar{\sigma}_{ij} \\ \omega_\varphi &\rightarrow \omega_\varphi + \lambda \omega_1 \\ \omega_\psi &\rightarrow \omega_\psi + \lambda \omega_2 \end{aligned} \quad (57)$$

for a fixed set of $U(1)^2$ -invariant functions $U_1, \bar{\sigma}_{ij}, \omega_1, \omega_2$, and small λ , and then seek a solution Φ of the form

$$\Phi = \Phi_0 + u. \quad (58)$$

where u is a function to be determined. Inserting (57) and (58) into (56), we have

$$\mathcal{E}(\lambda, u) = 0. \quad (59)$$

where

$$\mathcal{E}(\lambda, u) = \Delta_4(\Phi_0 + u) - \frac{1}{6}\tilde{R}_\lambda(\Phi_0 + u) + \frac{\tilde{K}_\lambda^2}{6(\Phi_0 + u)^5} \quad (60)$$

where \tilde{R}_λ and \tilde{K}_λ^2 are obtained from \tilde{R}_0 and \tilde{K}_0 using the transformation (57). If we plug in $\lambda = 0$ in equation (60), we have equation (56). Then to prove theorem 3.1, it is enough to show existence and uniqueness of the solution of equation (59) and this will be done in the next lemma. We then obtain a family of solutions $(\Sigma, h_{ab}^\lambda, K_{ab}^\lambda)$ to (47) and (48) with $h_{ab}^\lambda = \Phi^2 \tilde{h}_{ab}^\lambda$ and $K_{ab}^\lambda = \Phi^{-2} \tilde{K}_{ab}^\lambda$. \square

Lemma 4.1. Let $\omega_1, \omega_2 \in C_c^\infty(\mathbb{R}^4 \setminus \Gamma)$ and $U_1, \bar{\sigma}_{ij} \in C_c^\infty(\mathbb{R}^4 \setminus \{0\})$. Then, there exists $\lambda_0 > 0$ such that for all $\lambda \in (-\lambda_0, \lambda_0)$

1. There exists a solution $u(\lambda)$ of (59) belonging to $W_{-1}'^{2,3}$. (for clarify we suppress the r - and θ - dependence of $u(\lambda)$).
2. $u(\lambda)$ is continuously differentiable in λ and $\Phi_0 + u(\lambda) > 0$.
3. $u(\lambda)$ is the unique solution of (59) for small u and small λ .

Remark 4.1. Here $\Gamma \equiv \{\rho = 0\}$ is the axis on which the Killing part of (42) becomes degenerate (i.e. at least one combination of $\partial/\partial\psi$ and $\partial/\partial\phi$ vanishes).

Here $W_{-1}'^{2,3}$ is one of Bartnik's weighted Sobolev spaces (appendix A). This space is consistent with the desired fall-off conditions of the solution u at the cylindrical end and asymptotically flat end. Moreover, we do not expect u to be regular at the origin. By section 2.2 we know the angular momenta are equal to the difference to potentials evaluated on the endpoints of the axis (parameterized here by θ). Therefore, with the requirement that $\omega_1, \omega_2 \in C_c^\infty(\mathbb{R}^4 \setminus \Gamma)$ the angular momenta are preserved by the family of deformations, which implies part 2 of Theorem 3.1.

4.1 Proof of Lemma

The main tool we use to establish the Lemma is the implicit function theorem (see Appendix B of [5]). The argument closely parallels that given in [5] and proceeds as follows. Firstly, we select appropriate Banach spaces X, Y , and Z as required for the implicit function theorem. Then we find neighbourhoods $O_x \subset X$ and $O_y \subset Y$ for

which the map $\mathcal{E} : O_x \times O_y \rightarrow Z$ is well-defined. Care must be given to select Banach spaces that satisfy the fall-off conditions on the functions U , σ_{ij} , Φ_0 , ω_φ , and ω_ψ at infinity and singular behavior at the origin of the function Φ_0 . Since the solution need not be regular at the origin (we are working on $\mathbb{R}^4 - \{0\}$) we cannot select the standard weighted Sobolev spaces $W_{-1}^{2,3}$. To begin we verify that $\mathcal{E} : O_x \times O_y \rightarrow Z$ is C^1 . Next we show that $D_2\mathcal{E}(0,0)$ (which is defined in equation (70)) is an isomorphism between Y and Z . The implicit function theorem is then used to conclude the existence of a unique u with the properties of the lemma.

4.1.1 \mathcal{E} is well-defined

We choose $X = \mathbb{R}$, $Y = W_{-1}'^{2,3}$ and $Z = L_{-3}'^3$. Moreover, we choose $O_x = \mathbb{R}$ and $O_y = \{u \in W_{-1}'^{2,3} : \|u\|_{W_{-1}'^{2,3}} < \xi\}$ where ξ is computed as follows: by the inequality in Lemma A.1-3 for $u \in O_y$ we have

$$r|u| \leq C_0\xi. \quad (61)$$

where C_0 is a constant. Also by lemma B.2, we have

$$r\Phi_0 \geq (ab\mu)^{1/4}. \quad (62)$$

Then, if we choose ξ such that

$$\frac{(ab\mu)^{1/4}}{C_0} > \xi > 0, \quad (63)$$

then for all $u \in O_y$ we will have

$$0 < (ab\mu)^{1/4} - C_0\xi \leq r(\Phi_0 + u). \quad (64)$$

First we prove that $\mathcal{E} : \mathbb{R} \times O_y \rightarrow L_{-3}'^3$ is well-defined. That is, we need to show for $\lambda \in \mathbb{R}$ and $u \in O_y$ we have $\mathcal{E}(\lambda, u) \in L_{-3}'^3$. By using the triangle inequality for equation (59), we have

$$\begin{aligned} \|\mathcal{E}(\lambda, u)\|_{L_{-3}'^3} &\leq \underbrace{\|\Delta_4 u\|_{L_{-3}'^3}}_I + \underbrace{\|\Delta_4 \Phi_0\|_{L_{-3}'^3}}_{II} + \underbrace{\frac{1}{6} \|\tilde{R}_\lambda(\Phi_0 + u)\|_{L_{-3}'^3}}_{III} + \underbrace{\left\| \frac{\tilde{K}_\lambda^2}{6(\Phi_0 + u)^5} \right\|_{L_{-3}'^3}}_{IV} \end{aligned} \quad (65)$$

We will show each of these terms are bounded in $L_{-3}'^3$. To show this we will need the required properties of the functions ω_1, ω_2, U_1 and $\bar{\sigma}_{ij}$, as well as the particular fall-off conditions on functions (i.e U, σ'_{ij}) of the conformal Myers-Perry metric.

(I) Since $u \in O_y$

$$\|\Delta_4 u\|_{L_{-3}'^3} \leq \|u\|_{W_{-1}'^{2,3}} \leq C \quad (66)$$

where C is function of a and b . Henceforth, the notation C is a constant related only on metric parameters, i.e. a and b .

(II) In second term we use the bound on the Laplace operator lemma C.1-3:

$$\|\Delta_4 \Phi_0\|_{L'_{-3}} \leq \left\| \frac{C}{r^6} \right\|_{L'_{-3}} \leq C \quad (67)$$

Finally, since ω_1 and ω_2 have compact support outside the axis and U_1 and $\bar{\sigma}_{ij}$ have compact support outside the origin, and by using (64) and lemma B.3 one can show that (III) and (IV)) are bounded. The details are tedious and we omit them here. Thus $\mathcal{E} : \mathbb{R} \times O_y \rightarrow L'_{-3}$ is well-defined.

4.1.2 \mathcal{E} is C^1

We denote by $D_1 \mathcal{E}(\lambda, u)$ the partial Fréchet derivative of \mathcal{E} with respect to the first argument evaluated at (λ, u) and by $D_2 \mathcal{E}(\lambda, u)$ the partial Fréchet derivative of \mathcal{E} with respect to the second argument u . These operators are formally obtained by directional derivatives of \mathcal{E} and they are linear operators between the following spaces:

$$D_1 \mathcal{E}(\lambda, u) : \mathbb{R} \rightarrow L'_{-3}, \quad (68)$$

$$D_2 \mathcal{E}(\lambda, u) : W_{-1}'^{2,3} \rightarrow L'_{-3}. \quad (69)$$

We use the notation $D_1 \mathcal{E}(\lambda, u)[\zeta] \in L'_{-3}$ to denote the operator $D_1 \mathcal{E}(\lambda, u)$ acting on $\zeta \in \mathbb{R}$. Similarly, $D_2 \mathcal{E}(\lambda, u)[v] \in L'_{-3}$ denotes the operator $D_2 \mathcal{E}(\lambda, u)$ acting on $v \in W_{-1}'^{2,3}$. These linear operators will be

$$\begin{aligned} D_1 \mathcal{E}(\lambda, u)[\zeta] &= \frac{d}{dt} \mathcal{E}(\lambda + t\zeta, u)|_{t=0} = \frac{1}{6} \left(-D_1 \tilde{R}_\lambda(\Phi_0 + u) + \frac{D_1 \tilde{K}_\lambda^2}{(\Phi_0 + u)^5} \right) \zeta, \\ D_2 \mathcal{E}(\lambda, u)[v] &= \frac{d}{dt} \mathcal{E}(\lambda, u + tv)|_{t=0} = \Delta_4 v - \frac{1}{6} \left(\tilde{R}_\lambda + \frac{5\tilde{K}_\lambda^2}{(\Phi_0 + u)^6} \right) v \end{aligned} \quad (70)$$

Now, we will prove that the map $\mathcal{E} : \mathbb{R} \times O_y \rightarrow L'_{-3}$ is C^1 . As a result of the properties of functions of the metric, we cannot use the chain rule. Alternatively, we will show that:

1. The linear operator $D_1 \mathcal{E}(\lambda, u)[\zeta]$ and $D_2 \mathcal{E}(\lambda, u)[v]$ are bounded. i.e.

$$\|D_1 \mathcal{E}(\lambda, u)[\zeta]\|_{L'_{-3}} \leq C|\zeta|, \quad (71)$$

$$\|D_2 \mathcal{E}(\lambda, u)[v]\|_{L'_{-3}} \leq C \|v\|_{W_{-1}'^{2,3}}. \quad (72)$$

2. The linear operator $D_1\mathcal{E}(\lambda, u)[\zeta]$ and $D_2\mathcal{E}(\lambda, u)[v]$ are continuous in (λ, u) in the operator norms. That is, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\lambda_1 - \lambda_2| < \delta \implies \|D_1\mathcal{E}(\lambda_1, u) - D_1\mathcal{E}(\lambda_2, u)\|_{B(X, Z)} < \epsilon, \quad (73)$$

$$\|u_1 - u_2\|_{W_{-1}'^{2,3}} < \delta \implies \|D_2\mathcal{E}(\lambda, u_1) - D_2\mathcal{E}(\lambda, u_2)\|_{B(Y, Z)} < \epsilon. \quad (74)$$

3. The operators $D_1\mathcal{E}(\lambda, u)[\zeta]$ and $D_2\mathcal{E}(\lambda, u)[v]$ are the partial Fréchet derivatives of \mathcal{E} . That is

$$\lim_{\zeta \rightarrow 0} \frac{\|\mathcal{E}(\lambda + \zeta, u) - \mathcal{E}(\lambda, u) - D_1\mathcal{E}(\lambda, u)[\zeta]\|_{L_{-3}'^3}}{|\zeta|} = 0, \quad (75)$$

$$\lim_{v \rightarrow 0} \frac{\|\mathcal{E}(\lambda, u + v) - \mathcal{E}(\lambda, u) - D_2\mathcal{E}(\lambda, u)[v]\|_{L_{-3}'^3}}{\|v\|_{W_{-1}'^{2,3}}} = 0. \quad (76)$$

1. To prove inequality (71) we use triangle inequality, lemma B.3, and inequality (64) then

$$\begin{aligned} \|D_1\mathcal{E}(\lambda, u)[\zeta]\|_{L_{-3}'^3} &\leq \frac{|\zeta|}{6} \|D_1\tilde{R}_\lambda(\Phi_0 + u)\|_{L_{-3}'^3} + \frac{|\zeta|}{6} \left\| \frac{D_1\tilde{K}_\lambda^2}{(\Phi_0 + u)^5} \right\|_{L_{-3}'^3} \\ &\leq C |\zeta| \end{aligned} \quad (77)$$

similarly, by definition of O_y and lemma (B.3) we have

$$\begin{aligned} \|D_2\mathcal{E}(\lambda, u)[v]\|_{L_{-3}'^3} &\leq \|\Delta_4 v\|_{L_{-3}'^3} + \frac{1}{6} \|\tilde{R}_\lambda v\|_{L_{-3}'^3} + \left\| \frac{5\tilde{K}_\lambda^2}{6(\Phi_0 + u)^6} v \right\|_{L_{-3}'^3} \\ &\leq C \|v\|_{W_{-1}'^{2,3}}. \end{aligned} \quad (78)$$

2. To show $D_1\mathcal{E}(\lambda, u)$ is continuous (it is in fact uniformly continuous), we use the triangle inequality, inequality (64), and lemma (B.3). Then

$$\begin{aligned} \|D_1\mathcal{E}(\lambda_1, u) - D_1\mathcal{E}(\lambda_2, u)\|_{L_{-3}'^3} &\leq \frac{1}{6} \|(D_1\tilde{R}_{\lambda_1} - D_1\tilde{R}_{\lambda_2})(\Phi_0 + u)\|_{L_{-3}'^3} \\ &\quad + \left\| \frac{D_1\tilde{K}_{\lambda_1}^2 - D_1\tilde{K}_{\lambda_2}^2}{6(\Phi_0 + u)^5} \right\|_{L_{-3}'^3} \\ &\leq C |\lambda_1 - \lambda_2|. \end{aligned} \quad (79)$$

To prove continuity in u consider the following identity for arbitrary x, y and integer p :

$$\frac{1}{x^p} - \frac{1}{y^p} = (y - x) \sum_{i=0}^{p-1} x^{i-p} y^{-1-i}. \quad (80)$$

Then

$$r^{-7} \left(\frac{1}{(\Phi_0 + u_1)^6} - \frac{1}{(\Phi_0 + u_2)^6} \right) = (u_2 - u_1) M. \quad (81)$$

where

$$M = \sum_{i=0}^5 (r(u + \Phi_0))^{i-6} (r\Phi_0)^{-1-i}. \quad (82)$$

Since $u_1, u_2 \in O_y$, and using the lower bound in equation (64) we have

$$M \leq C. \quad (83)$$

Then by (83) and Lemma B.2-2 we have

$$\begin{aligned} \|D_2\mathcal{E}(\lambda, u_1)[v] - D_2\mathcal{E}(\lambda, u_2)[v]\|_{L'^3_{-3}} &= \left\| v \frac{5\tilde{K}_\lambda^2}{6(\Phi_0 + u_1)^6} - v \frac{5\tilde{K}_\lambda^2}{6(\Phi_0 + u_2)^6} \right\|_{L'^3_{-3}} \\ &\leq C \left\| \frac{(u_1 - u_2)v}{r} \right\|_{L'^3_{-3}} \end{aligned} \quad (84)$$

The right hand side of the above equation can be bounded as follows: (we write dx to represent the volume element for the Euclidean metric on $\mathbb{R}^4 \setminus \{0\}$)

$$\begin{aligned} \left\| \frac{(u_1 - u_2)v}{r} \right\|_{L'^3_{-3}} &= \left(\int_{\mathbb{R}^4 \setminus \{0\}} \frac{(u_1 - u_2)^3 v^3}{r^3} r^5 dx \right)^{1/3} \\ &= \left(\int_{\mathbb{R}^4 \setminus \{0\}} \frac{(u_1 - u_2)^3 (rv)^3}{r} dx \right)^{1/3} \\ &\leq C \|v\|_{W'^{2,3}_{-1}} \left(\int_{\mathbb{R}^4 \setminus \{0\}} \frac{(u_1 - u_2)^3}{r} dx \right)^{1/3} \\ &\leq C \|v\|_{W'^{2,3}_{-1}} \|u_1 - u_2\|_{W'^{2,3}_{-1}}. \end{aligned} \quad (85)$$

The first inequality follows from Lemma A.1 and the second inequality from the definition of Sobolev norms. Therefore, we have

$$\|D_2\mathcal{E}(\lambda, u_1)[v] - D_2\mathcal{E}(\lambda, u_2)[v]\|_{L'^3_{-3}} \leq C \|v\|_{W'^{2,3}_{-1}} \|u_1 - u_2\|_{W'^{2,3}_{-1}}. \quad (86)$$

Thus, $D_2G(\lambda, u)$ is a continuous operator.

3. Equation (75) is straightforward to prove. We prove (76) as follows

$$\mathcal{E}(\lambda, u + v) - \mathcal{E}(\lambda, u) - D_2\mathcal{E}(\lambda, u)[v] = \frac{\tilde{K}_\lambda^2}{6} \left(\frac{1}{(\Phi_0 + u + v)^5} - \frac{1}{(\Phi_0 + u)^5} + \frac{5v}{(\Phi_0 + u)^6} \right)$$

By simplifying we have

$$r^{-7} \left(\frac{1}{(\Phi_0 + u + v)^5} - \frac{1}{(\Phi_0 + u)^5} + \frac{5v}{(\Phi_0 + u)^6} \right) = v^2 M_1. \quad (87)$$

where

$$M_1 = \frac{1}{(r(\Phi_0 + u + v))^5 (r(\Phi_0 + u))^6} \sum_{\substack{i+j+k=4 \\ \forall i,j,k \geq 0}} C_{ijk} (r\Phi_0)^i (ru)^j (rv)^k. \quad (88)$$

Where C_{ijk} are numerical constants. To find the bound of M_1 we will use equation (61) and the fact that $u, v \in V$. Then we have

$$|M_1| \leq C \frac{(r^2 + ab + b^2)(r^2 + ab + a^2) + \mu^2}{\left([(r^2 + ab + b^2)(r^2 + ab + a^2) + \mu^2]^{1/4} - C_0 \xi \right)^{11}} \leq C. \quad (89)$$

Then by lemma B.3 and above inequality we have

$$\|\mathcal{E}(\lambda, u + v) - \mathcal{E}(\lambda, u) - D_2 \mathcal{E}(\lambda, u)[v]\|_{L'^3_{-3}} \leq C \left\| \frac{v^2 M_1}{r} \right\|_{L'^3_{-3}} \leq C \|v\|_{W'^{2,3}_{-1}}^2. \quad (90)$$

By steps similar to (85) we have second inequality. Hence, we have proved statements (1),(2), and (3) and $\mathcal{E}(\lambda, u) : \mathbb{R} \times O_y \rightarrow L'^3_{-3}$ is C^1 .

4.1.3 $D_2 \mathcal{E}(0, 0)$ is an isomorphism

We now verify $D_2 \mathcal{E}(0, 0) : W'^{2,3}_{-1} \rightarrow L'^3_{-3}$ is an isomorphism. We write this linear operator as

$$D_2 \mathcal{E}(0, 0)[v] = \Delta_4 v - \alpha v \quad (91)$$

where

$$\alpha = \frac{\tilde{R}_0}{6} + \frac{5\tilde{K}_0^2}{6\Phi_0^6} \quad (92)$$

An important property of the function α by lemma B.1 is a nonnegative bounded function in $\mathbb{R}^4 \setminus \{0\}$, that is $\alpha = hr^{-6}$ where $h \geq 0$. Therefore $\alpha \in L'^3_{-3}$. Hence, as shown in Appendix A, when $M = \mathbb{R}^4 \setminus \{0\}$ and $p = 3, \delta = -1$, the map $\Delta_4 - \alpha$ is an isomorphism from $W'^{2,3}_{-1} \rightarrow L'^3_{-3}$.

5 Discussion

We have constructed a one-parameter family of initial data to the vacuum Einstein's equations with the same symmetries and asymptotic behaviour as initial data for the extreme Myers-Perry black hole in five dimensions. In particular this data have the

same angular momenta (J_1, J_2) . Such initial data will generically have a non-stationary evolution and is a starting point to investigate the dynamics near extremality for such black holes. Our results generalize the analogous results concerning initial data ‘close’ to extreme Kerr data [5]. An important property of this three-dimensional initial data is that they had strictly greater energy than extreme Kerr. This is a consequence of Dain’s mass-angular momentum inequality, valid in axisymmetry: $M \geq \sqrt{J}$, for which the initial data for extreme Kerr is the unique minimizer that saturates the bound [11, 12, 31]. In our case, however, in the absence of geometric inequalities we cannot conclude that the energy of the family of initial data discussed is strictly greater than that of extreme Myers-Perry. Noting that the mass of Myers-Perry black holes satisfy the bound

$$M^3 \geq \frac{27\pi}{32} (|J_1| + |J_2|)^2 \quad (93)$$

with equality in the extreme case, it would be tempting to conjecture Dain’s inequality admits a generalization to four-dimensional biaxisymmetric initial data. Proving that the extreme Myers-Perry initial data is local minimizer of energy amongst the class of initial data we have considered here would be a useful first step towards establishing an analogue of Dain’s global result. Note, however, that the energy of an extreme black ring [34] satisfies

$$M^3 = \frac{27\pi}{4} |J_1| (|J_2| - |J_1|) \quad (94)$$

This suggests a more complicated geometric inequality in five dimensions, which takes into account which combination of rotational Killing fields have fixed points in the interior of the initial data. We hope to address these issues in future work.

The method used here to find solutions of the constraint equations relied on the ability to generate initial data from the specification of scalar functions and reduce the problem to a single scalar PDE. In particular, the assumption of ‘ $t - \phi^i$ ’ symmetry allows one to determine the extrinsic curvature completely from the twist potentials. The existence of these potentials in turn relied on the existence of $U(1)^2$ isometry. It is clear that this technique would work in spacetime dimensions $D > 5$, provided one assumes $U(1)^{D-3}$ isometry. Of course, this is too much Abelian symmetry to describe an asymptotically flat black hole for $D > 5$. However, in certain limits extra Abelian symmetry may arise. For example, for higher-dimensional Myers-Perry black holes, one may take an ‘ultraspinning limit’ which enhances the number of commuting isometries (the limit changes the black hole horizon from S^{D-2} to $S^p \times \mathbb{T}^q$ for appropriately chosen integers (p, q)) [35]. It is known that *non-extremal* black holes with a single non-zero angular momentum admit a linearized gravitational instability in the ultraspinning limit [36]. It might be interesting to investigate ultraspinning instabilities of *extremal* black holes in $D > 5$ using the formalism described here. The initial data under consideration would have, in addition to a cylindrical end, an asymptotically Kaluza-Klein end, rather than an asymptotically flat one.

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A Asymptotically Euclidean manifolds

A precise mathematical formalism to describe the asymptotic behaviour of functions on a space is the theory of weighted Sobolev spaces. Here we use Bartnik's weighted Sobolev space [26, 6] which is appropriate for Riemannian manifolds with asymptotically Euclidean and cylindrical ends. The weight function is $r = |x|$ for $x \in \mathbb{R}^n$. Then for any $\delta \in \mathbb{R}$, $1 \leq p < \infty$, Bartnik's weighted Sobolev space $W'_\delta{}^{k,p}$ is the subset of $W'_{\text{loc}}{}^{k,p}$ for which the norm

$$\|u\|_{W'_\delta{}^{k,p}} = \sum_{j=0}^k \|\partial^j u\|_{L'_{\delta^{-j}}{}^p}, \quad \text{where} \quad \|u\|_{L'_\delta{}^p} = \left(\int_{\mathbb{R}^n \setminus \{0\}} |u|^p r^{-\delta p - n} dx \right)^{1/p} \quad (95)$$

is finite. Relevant properties of this weighted Sobolev space are summarized in the following lemma [26, 6, 5]

Lemma A.1. :

1. If $p \leq q$ and $\delta_1 < \delta_2$ then $L'_{\delta_1}{}^p \subset L'_{\delta_2}{}^q$ and the inclusion is continuous.
2. For $k \geq 1$ and $\delta_1 < \delta_2$ the inclusion $W'_{\delta_1}{}^{k,p} \subset W'_{\delta_2}{}^{k-1,p}$ is compact.
3. If $1/p < k/n$ then $W'_\delta{}^{k,p} \subset C_\delta'^0$. The inclusion is continuous. That is if $u \in W'_\delta{}^{k,p}$ then $r^{-\delta} |u| \leq C \|u\|_{W'_\delta{}^{k,p}}$. Further, as proved in [5], $\lim_{r \rightarrow 0} r^{-\delta} |u| = \lim_{r \rightarrow \infty} r^{-\delta} |u| = 0$.

Let M be a smooth, connected, complete, n -dimensional Riemannian manifold (M, γ) , and let $\rho < 0$. We say (M, γ) is asymptotically Euclidean of class $W'_\rho{}^{k,p}$ if

- The metric $\gamma \in W'_\rho{}^{k,p}(M)$, where $1/p - k/n < 0$ and γ is continuous.
- There exists a finite collection $\{N_i\}_{i=1}^m$ of open subsets of M and diffeomorphisms $\Phi_i : E_r \rightarrow N_i$ ($E_r = \mathbb{R}^n \setminus \bar{B}_r(0)$) such that $M - \cup_i N_i$ is compact.
- For each i , $\Phi_i^* \gamma - \bar{\gamma} \in W'_\rho{}^{k,p}(E_r)$

We call the charts Φ_i end charts and the corresponding coordinates are end coordinates. Now, suppose (M, γ) is asymptotically Euclidean, and let $\{\Phi_i\}_{i=1}^m$ be its collection of end charts. Let $K = M - \cup_i \Phi_i(E_{2r})$, so K is a compact manifold. The weighted Sobolev space $W_\delta^{k,p}(M)$ is the subset of $W_{\text{loc}}^{k,p}(M)$ such that the norm

$$\|u\|_{W_\delta^{k,p}(M)} = \|u\|_{W^{k,p}(K)} + \sum_i \|\Phi_i^* u\|_{W_\delta^{k,p}(E_r)} \quad (96)$$

is finite. We can define similarly weighted Lebesgue space $L_\delta^p(M)$ and $C_\delta'^k$ and $C_\delta'^\infty(M) = \cap_{k=0}^\infty C_\delta'^k(M)$. In the particular case when $M = \mathbb{R}^n$, then we have just one asymptotically Euclidean end. Moreover, if (M, γ) is an asymptotically Euclidean manifold of class $W_\rho'^{k,p}$, we say (M, γ, K) is asymptotically Euclidean dataset if $K \in W_{\rho-1}'^{k-1,p}(M)$.

The main goal of this appendix is to consider the Poisson operator $\mathcal{L} = \Delta_\gamma - \alpha$ on scalar functions of an asymptotically Euclidean manifold and express a very classical result ([37] or see [6]), that is, \mathcal{L} is an isomorphism from Sobolev space $W_\delta'^{2,p}$ to L_δ^p . We start with the estimate [38, 6, 1]

Lemma A.2. Suppose (M, γ) is asymptotically Euclidean of class $W_\rho'^{2,p}$, $p > \frac{n}{2}$, $\rho < 0$. Then if $2 - n < \delta < 0$, $\delta' \in \mathbb{R}$, and $u \in W_\delta'^{2,p}$ we have

$$\|u\|_{W_\delta'^{2,p}} \leq \|\mathcal{L}u\|_{L_{\delta-2}^p} + \|u\|_{L_{\delta'}^p}. \quad (97)$$

Now we have following weak maximum principle (Lemma 3.2 in [6])

Lemma A.3. Suppose (M, γ) is asymptotically Euclidean of class $W_\rho'^{k,p}$, $k \geq 2$, $k > \frac{n}{p}$, and suppose $\alpha \in W_{\rho-2}'^{k-2,p}$ and suppose $\alpha \geq 0$. If $u \in W_{\text{loc}}'^{k,p}$ satisfies

$$-\Delta_\gamma u + \alpha u \leq 0 \quad (98)$$

and if $u^+ \equiv \max(u, 0)$ is $o(1)$ on each end of M , then $u \leq 0$. In particular, if $u \in W_\delta'^{k,p}$ for some $\delta < 0$ and u satisfies (98), then $u \leq 0$.

Proof. Fix $\epsilon > 0$, and let $v = (u - \epsilon)^+$. Since $u^+ = o(1)$ on each end, we see v is compactly supported. Moreover, since $u \in W_{\text{loc}}'^{k,p}$ we have from Sobolev embedding that $u \in W_{\text{loc}}'^{1,2}$ and hence $v \in W'^{1,2}$. Now,

$$\int_M (-v \Delta_\gamma u + \alpha uv) \, dx \leq 0 \implies \int_M -v \Delta_\gamma u \, dx \leq - \int_M \alpha uv \, dx \leq 0 \quad (99)$$

where dx denotes the volume element on (M, γ) . Since $\alpha \geq 0$, $v \geq 0$ and u is positive wherever $v \neq 0$. Integrating by parts we have

$$\int_M |\nabla v|^2 \, dx \leq 0 \quad (100)$$

since $\nabla u = \nabla v$ on the support of v . So v is constant and compactly supported, so it should be zero, i.e. $\max(u - \epsilon, 0) = 0$. Then we conclude $u \leq \epsilon$. Sending ϵ to 0 we have $u \leq 0$.

Now, if $u \in W_\delta'^{k,p}$, since $W_\delta'^{k,p} \subset C_\delta'^0$, we have $u \in C_\delta'^0$. Hence if $\delta < 0$, then $u^+ = o(1)$ and lemma can be applied to u . \square

Using this Lemma we can prove the following theorem.

Theorem A.1. Suppose (M, γ) is asymptotically Euclidean of class $W_\rho'^{2,p}$, $p > \frac{n}{2}$. Then if $2 - n < \delta < 0$ and $\alpha \in L_{\delta-2}^p$, the operator $\mathcal{L} : W_\delta'^{2,p} \rightarrow L_{\delta-2}^p$ is Fredholm with index 0. Moreover, if $\alpha \geq 0$ then \mathcal{L} is an isomorphism.

Proof. By the estimate in Lemma A.2 and [38] this operator is Fredholm. Now we show \mathcal{L} is injective. Let $\mathcal{L}u = 0$ for $u \in W_\delta'^{2,p}$. Then by weak maximum principle we have $u = 0$ on M for $2 - n < \delta < 0$ and \mathcal{L} is injective. To show \mathcal{L} is surjective, it suffices to show \mathcal{L}^* is injective from $L_{2-n-\delta}^p \rightarrow W_{-n-\delta}'^{-2,p}$. Now let f_1 and f_2 be smooth and compactly supported in each end of M . We have from integration by parts

$$0 = \langle f_2, \mathcal{L}^*(f_1) \rangle = \langle \mathcal{L}(f_2), f_1 \rangle = \int_M \mathcal{L}(f_2) f_1 dx \quad (101)$$

Thus $\int_M \mathcal{L}(f_2) f_1 dx = 0$ for all smooth and compactly supported f_2 in each end of M , then $f_1 = 0$ and \mathcal{L}^* is injective. Then \mathcal{L} is surjective. Therefore, \mathcal{L} is an isomorphism. \square

B Myers-Perry black hole initial data

In this Appendix we will give details on various properties of the initial data for the extreme Myers-Perry metric. We have used MAPLE to simplify a number of our computations. Our main interest is to find certain final bounds and since most of the calculations are similar, we only provide explicit details for a subset of cases. The slice metric can be written as

$$h = \frac{\Sigma}{r^2} (dr^2 + r^2 d\theta^2) + \sigma_{ij} d\phi^i d\phi^j \quad (102)$$

where

$$\begin{aligned} \Sigma &= r^2 + ab + a^2 \cos^2 \theta + b^2 \sin^2 \theta & \sigma_{12} &= \frac{ab\mu}{\Sigma} \sin^2 \theta \cos^2 \theta \\ \sigma_{11} &= \frac{a^2\mu}{\Sigma} \sin^4 \theta + (r^2 + ab + a^2) \sin^2 \theta & \sigma_{22} &= \frac{b^2\mu}{\Sigma} \cos^4 \theta + (r^2 + ab + b^2) \cos^2 \theta \end{aligned} \quad (103)$$

where $\phi^1 = \varphi$ and $\phi^2 = \psi$. Now if we choose $\rho = \frac{1}{2}r^2 \sin 2\theta$ and $z = \frac{1}{2}r^2 \cos 2\theta$, then the conformal slice metric of the extreme Myers-Perry black hole can be written

$$\tilde{h} = e^{2U} (d\rho^2 + dz^2) + \sigma'_{ij} d\phi^i d\phi^j \quad (104)$$

where

$$\Phi_0^2 = \frac{\sqrt{\det \sigma}}{\rho} \quad \sigma'_{ij} = \Phi_0^{-2} \sigma_{ij} \quad e^{2U} = \Phi_0^{-2} \frac{\Sigma}{r^4} \quad (105)$$

In general, the lapse and shift vectors are degrees of freedom for the initial data set. But since we want to preserve geometrical properties of the initial data under evolution, we compute the lapse of the extreme Myers-Perry spacetime and select the shift vector to be the product of r and the shift of extreme Myers-Perry metric.

$$\alpha = \sqrt{\frac{r^4 \Sigma}{(\Sigma + \mu)r^4 + \mu^2(r^2 + ab)}}, \quad (106)$$

$$\beta^\varphi = \frac{ra\mu(r^2 + ab + b^2)}{(\Sigma + \mu)r^4 + \mu^2(r^2 + ab)}, \quad \beta^\psi = \frac{rb\mu(r^2 + ab + a^2)}{(\Sigma + \mu)r^4 + \mu^2(r^2 + ab)} \quad (107)$$

In addition, we showed in section 2.2 that the extrinsic curvature can be generated from scalar potentials ω_{ϕ^i} . In the coordinate system used above, these are

$$\omega_\varphi = \frac{a(a^2 - b^2)(r^2 + ab + b^2) \cos^2 \theta - r^2 a(2a^2 + 2ab + r^2)}{(a - b)^2} + \frac{a(r^2 + ab + a^2)^2(r^2 + ab + b^2)}{\Sigma(a - b)^2} \quad (108)$$

$$\omega_\psi = \frac{br^2((a + b)^2 + r^2) - b(a^2 - b^2)(r^2 + ab + a^2) \cos^2 \theta}{(a - b)^2} - \frac{b(r^2 + ab + a^2)(r^2 + ab + b^2)^2}{\Sigma(a - b)^2} \quad (109)$$

It is important to mention

$$\Delta_2 = \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^4} \left(r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} \right) \quad (110)$$

Now we will prove some useful lemmas for the main theorem.

Lemma B.1. The function α in equation (92) is nonnegative and has following bounds

$$\alpha = \frac{\tilde{K}_0^2}{2\Phi_0^6} + r^2(dv)^2 = hr^{-6} \quad (111)$$

where h is a bounded nonnegative function.

Proof. First we know by conformal transformation $h_{ab} = \Phi^2 \tilde{h}_{ab}$ the scalar curvature will be²

$$\tilde{R} = R\Phi^2 + 6 \left(\Delta_{\tilde{h}} v + |\tilde{\nabla} v|^2 \right) \quad (112)$$

where $v = \log \Phi$. By constraint equations (47) and the fact that conformal extreme Myers-Perry satisfies in relation

$$\Delta_{\tilde{h}} v = -\frac{1}{2\Phi^6} \tilde{K}_{ab} \tilde{K}^{ab} \quad (113)$$

²There are some typos about factors in journal version.

we have

$$\begin{aligned}\tilde{R} &= K_{ab}K^{ab}\Phi^2 - 3\Phi^{-6}\tilde{K}_{ab}\tilde{K}^{ab} + 6|\tilde{\nabla}v|^2 \\ &= -2\tilde{K}_{ab}\tilde{K}^{ab}\Phi^{-6} + 6e^{-2U}(dv)^2\end{aligned}\quad (114)$$

Then by equations (54) and (55) we have

$$\tilde{R}_0 = -5\tilde{K}_0^2\Phi^{-6} + 6r^2(dv)^2 \quad (115)$$

Therefore, α is

$$\alpha = \frac{\tilde{R}_0}{6} + \frac{5\tilde{K}_0^2}{6\Phi_0^6} = \frac{\tilde{K}_0^2}{2\Phi_0^6} + r^2(dv)^2 = hr^{-6} \quad (116)$$

□

Lemma B.2. Let Φ_0 , \tilde{R}_0 , and \tilde{K}_0^2 be defined as in (105), (54), and (56), respectively. Then we have following bounds:

1. $(ab\mu)^{1/4} \leq [(r^2 + ab + b^2)(r^2 + ab + a^2)]^{1/4} \leq r\Phi_0 \leq [(r^2 + ab + b^2)(r^2 + ab + a^2) + \mu^2]^{1/4}$
2. $|\tilde{R}_0| \leq \frac{C}{r^4}$ and $|\tilde{K}_0^2| \leq \frac{C}{r^6}$
3. $|\Delta_4\Phi_0| \leq \frac{C}{r^6}$

Proof. We will prove just 1 here; the remaining bounds require lengthy algebraic manipulations.

1. We have

$$\begin{aligned}r^2\Phi_0^2 &= \left[(r^2 + ab + b^2)(r^2 + ab + a^2) + \frac{\mu(r^2 + ab)(a^2 \cos^2 \theta + b^2 \sin^2 \theta) + \mu a^2 b^2}{\Sigma} \right]^{1/2} \\ &\leq [(r^2 + ab + b^2)(r^2 + ab + a^2) + \mu^2]^{1/2}\end{aligned}\quad (117)$$

so if $r \rightarrow \infty$ then we have minimum of $r^2\Phi_0^2$

$$\sqrt{(r^2 + ab + b^2)(r^2 + ab + a^2)} \leq r^2\Phi_0^2 \quad (118)$$

Therefore for $a, b > 0$ we have

$$(ab\mu)^{1/4} \leq [(r^2 + ab + b^2)(r^2 + ab + a^2)]^{1/4} \leq r\Phi_0 \leq [(r^2 + ab + b^2)(r^2 + ab + a^2) + \mu^2]^{1/4} \quad (119)$$

□

Lemma B.3. If we transform metric functions by (57) for small λ (i.e. $-\lambda_0 < \lambda < \lambda_0$) then

1. $\left\| \tilde{R}_\lambda \right\|_{L_{-3}^3} \leq C$
2. $\left\| \tilde{K}_\lambda^2 \right\|_{L_{-3}^3} \leq C$
3. $\left\| D_1 \tilde{R}_\lambda \right\|_{L_{-3}^3} \leq C$
4. $\left\| D_1 \tilde{K}_\lambda^2 \right\|_{L_{-3}^3} \leq C$
5. $\left\| D_1 \tilde{R}_{\lambda_1} - D_1 \tilde{R}_{\lambda_2} \right\|_{L_{-3}^3} \leq C |\lambda_1 - \lambda_2|$
6. $\left\| D_1 \tilde{K}_{\lambda_1}^2 - D_1 \tilde{K}_{\lambda_2}^2 \right\|_{L_{-3}^3} \leq C |\lambda_1 - \lambda_2|$

Proof. We will prove numbers 1 and 4 of these inequalities and others will be similar.

1) By definition of \tilde{R}_λ we have

$$\begin{aligned}
\tilde{R}_\lambda &= -r^2 \Delta_2 (U + \lambda U_1) + r^2 \frac{\det(d\sigma' + \lambda d\bar{\sigma})}{2\rho^2} \\
&= -r^2 \Delta_2 U - r^2 \lambda \Delta_2 U_1 + \frac{r^2}{2\rho^2} [(d\sigma'_{11} + \lambda d\bar{\sigma}_{11}) \cdot (d\sigma'_{22} + \lambda d\bar{\sigma}_{22}) - (d\sigma'_{12} + \lambda d\bar{\sigma}_{12})^2] \\
&= \tilde{R}_0 - r^2 \lambda \Delta_2 U_1 + \frac{r^2}{2\rho^2} [\lambda d\bar{\sigma}_{11} \cdot d\sigma'_{22} + \lambda (d\sigma'_{11} + \lambda d\bar{\sigma}_{11}) \cdot d\bar{\sigma}_{22} - \lambda (2d\sigma'_{12} + \lambda d\bar{\sigma}_{12}) \cdot d\bar{\sigma}_{12}]
\end{aligned} \tag{120}$$

Then by triangle inequality we have

$$\begin{aligned}
\left\| \tilde{R}_\lambda \right\|_{L_{-3}^3} &\leq \left\| \tilde{R}_0 \right\|_{L_{-3}^3} + |\lambda| \left\| r^2 \Delta_2 U_1 \right\|_{L_{-3}^3} + \left\| \frac{r^2}{2\rho^2} (\lambda d\bar{\sigma}_{11} \cdot d\sigma'_{22} + \lambda (d\sigma'_{11} + \lambda d\bar{\sigma}_{11}) \cdot d\bar{\sigma}_{22}) \right\|_{L_{-3}^3} \\
&\quad + \left\| \frac{r^2}{2\rho^2} (\lambda (2d\sigma'_{12} + \lambda d\bar{\sigma}_{12}) \cdot d\bar{\sigma}_{12}) \right\|_{L_{-3}^3} \\
&\leq C
\end{aligned} \tag{121}$$

We used inequality of Lemma B.2-2 and the fact that functions U_1 and $\bar{\sigma}_{ij}$ have compact support outside the origin.

4) By definition of full contraction of extrinsic curvature we have

$$\begin{aligned}
\tilde{K}_\lambda^2 &= \frac{r^2}{2\rho^4} \left[(\sigma'_{11} + \lambda \bar{\sigma}_{11})(d\omega_\psi + \lambda d\omega_2)^2 + (\sigma'_{22} + \lambda \bar{\sigma}_{22})(d\omega_\varphi + \lambda d\omega_1)^2 \right. \\
&\quad \left. - 2(\sigma'_{12} + \lambda \bar{\sigma}_{12})(d\omega_\psi + \lambda d\omega_2) \cdot (d\omega_\varphi + \lambda d\omega_1) \right]
\end{aligned} \tag{122}$$

Then have

$$\begin{aligned}
D_1 \tilde{K}_\lambda^2 &= \frac{r^2}{2\rho^4} \left[\bar{\sigma}_{11}(d\omega_\psi + \lambda d\omega_2)^2 + 2(\sigma'_{11} + \lambda \bar{\sigma}_{11})d\omega_2 \cdot (d\omega_\psi + \lambda d\omega_2) + \bar{\sigma}_{22}(d\omega_\varphi + \lambda d\omega_1)^2 \right. \\
&\quad + (\sigma'_{22} + \lambda \bar{\sigma}_{22})d\omega_1 \cdot (d\omega_\varphi + \lambda d\omega_1) - 2\lambda \bar{\sigma}_{12}(d\omega_\psi + \lambda d\omega_2) \cdot (d\omega_\varphi + \lambda d\omega_1) \\
&\quad \left. - 4(\sigma'_{12} + \lambda \bar{\sigma}_{12})\lambda d\omega_2 \cdot (d\omega_\varphi + \lambda d\omega_1) - 4(\sigma'_{12} + \lambda \bar{\sigma}_{12})(d\omega_\psi + \lambda d\omega_2) \cdot d\omega_1 \right]
\end{aligned} \tag{123}$$

Then by triangular inequality and the fact that ω_i has compact support outside axis and $\bar{\sigma}_{ij}$ has compact support outside the origin one can show it is bounded. \square

Finally, we have following limits

$$\lim_{s \rightarrow \infty} \omega_\varphi = \frac{ab\mu \cos^2 \theta}{(a-b)} + \frac{a^3 b \mu (a+b)}{(ab + a^2 \cos^2 \theta + b^2 \sin^2 \theta)(a-b)^2} \quad (124)$$

$$\lim_{s \rightarrow \infty} \omega_\psi = -\frac{ab\mu \cos^2 \theta}{(a-b)} - \frac{ab\mu(a+b)}{(ab + a^2 \cos^2 \theta + b^2 \sin^2 \theta)(a-b)^2} \quad (125)$$

$$\lim_{s \rightarrow \infty} r\Phi_0 = \frac{\mu(\mu - ab)}{2(ab + a^2 \cos^2 \theta + b^2 \sin^2 \theta)} . \quad (126)$$

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